π.χ.

$(1-x^2)y'' - xy' + y = 0 \quad // \quad x_0 = 0, \; y_0(0) = 2, \; y_0'(0) = 1$

$a_2(x) = 1-x^2$ | $a_2(0) = 1 \neq 0$

$a_1(x) = -x$ | $\dfrac{a_1(x)}{a_2(x)} = \dfrac{-x}{1-x^2} = -x\dfrac{1}{1-x^2} = -x\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty}(-1)x^{2n+1}$

$a_0(x) = 1$

$|x^2|<1 \Rightarrow R_1 = 1$

$\dfrac{a_0(x)}{a_2(x)} = \dfrac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}, \quad R_2 = 1$

$y(x) = \sum_{n=0}^{\infty} C_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} C_n n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$

Κάνω αντικατάσταση:

$0 = (1-x^2)\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} C_n n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n$

$= \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=1}^{\infty} C_n n x^n + \sum_{n=0}^{\infty} C_n x^n.$

$= \sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} C_n n(n-1)x^n - \sum_{n=1}^{\infty} C_n n x^n + \sum_{n=0}^{\infty} C_n x^n$

↳ τα αθροίζω στη μεγαλύτερη δύναμη

$= \left[C_2 \cdot 2\cdot 1 \cdot x^0 + C_3 \cdot 3 \cdot 2x + \sum_{n=2}^{\infty} C_{n+2}(n+2)(n+1)x^n\right] - \sum_{n=2}^{\infty} C_n n(n-1)x^n - C_1 \cdot 1x - \sum_{n=2}^{\infty} C_n n x^n + \sum_{n=2}^{\infty} C_n x^n + C_0 + C_1 x$

$= 2C_2 + 6C_3 x + C_0 + \cancel{C_1 x} + \sum_{n=2}^{\infty}\left[C_{n+2}(n+2)(n+1) - C_n n(n-1) - nC_n + C_n\right]x^n - \cancel{C_1 x} = 0$

$C_{n+2}(n+2)(n+1) - C_n\left[\underbrace{n(n-1)+n-1}_{n^2-n+n-1 \; = \; n^2-1}\right] = 0$

$$\begin{cases} C_0 + 2C_2 = 0 \Rightarrow C_2 = \frac{1}{2}C_0 \\ C_3 = 0 \\ C_{n+2} = \dfrac{(n-1)\cancel{(n+1)}}{\cancel{(n+1)}(n+2)} C_n, \; n \geq 2 \end{cases} \rightarrow \boxed{C_n = \frac{n-3}{n} C_{n-2}, \; n \geq 4}$$

Παίρνω 2 περιπτώσεις (άρτιους - περιττούς) γιατί έχω βήμα 2:

<u>n = 2k</u>

$C_{2k} = \dfrac{2k-3}{2k} C_{2(k-1)}, \quad k \geq 2$

$k=2: \quad C_4 = \dfrac{2\cdot 2-3}{2\cdot 2} C_2$

$k=3$: $C_6 = \frac{2\cdot 3-3}{2\cdot 3} C_4$

$k=4$: $C_8 = \frac{2\cdot 4-3}{2\cdot 4} C_6$

.....

$k=n$: $C_{2n} = \frac{2\cdot n-3}{2n} C_{2(n-1)}$

Άρα: $C_{2n} = \frac{(\cancel{2\cdot 2}-3)^{1}(\cancel{2\cdot 3}-3)^{3}(\cancel{2\cdot 4}-3)^{5}\cdots(2n-3)}{2^{n-1}\, n!} C_2 \Rightarrow$

$$\boxed{C_{2n} = \frac{1\cdot 3\cdots(2n-3)}{2^{n-1}\, n!}\left(-\frac{1}{2}\right) C_0 \ , n\geq 2}$$

$2k+1\geq 4 \Rightarrow 2k\geq 3 \Rightarrow k\geq \frac{3}{2}$

$\underline{n=2k+1}$, $k\geq 2$

$C_{2k+1} = \frac{2k+1-3}{2k+1} C_{2k-1}$, $k\geq 2$

$k=2$: $C_5 = \frac{2\cdot 2-2}{2\cdot 2+1} C_3$

$k=3$: $C_7 = \frac{2\cdot 3-2}{2\cdot 3+1} C_5$

.....

$C_{2n+1} = \frac{2n-2}{2n+1} C_{2n-1}$ Αλλά επειδή $C_3=0$, όλοι αυτοί οι όροι μηδενίζονται.

Δηλαδή, δεν υπάρχουν όροι περιττής τάξης εκτός από το C_1.

$y(x) = \sum_{n=0}^{\infty} x^n = C_0 + C_1 x + \sum_{n=2}^{\infty} C_n x^n$

$= C_0 + C_1 x + \sum_{n=2}^{\infty} \underbrace{C_{2n}}_{\text{άρτιοι όροι}} x^{2n} + \sum_{n=2}^{\infty} \underbrace{\cancel{C_{2n+1}}}_{\text{περιττοί όροι}}{}^{0} x^{2n+1}$

$= C_0 + C_1 x + \sum_{n=2}^{\infty} (\ldots) C_0 x^{2n}$

Άρα $y(x) = C_0 \underbrace{\overbrace{\left[1 + \sum_{n=2}^{\infty} (\ldots) x^{2n}\right]}^{\text{δεν ορίζεται πέρα από το } 1}}_{y_1(x)} + C_1 \underbrace{x}_{y_2(x)}$ είναι λύση της εξίσωσης που ορίζεται στο $\mathbb{R}$.

Π.Χ. (Airy)

$y'' - xy = 0 \quad (x_0 = 0) \quad , \quad R_1 = +\infty = R_2 \rightarrow \mathbb{R}$

$y(x) = \sum_{n=0}^{\infty} c_n x^n \quad (\ldots)$

$\rightarrow c_2 = 0$

$(n+2)(n+1)c_{n+2} - c_{n-1} = 0 \quad , \quad n = 1, 2, \ldots$

$c_{n+2} = \frac{1}{(n+1)(n+2)} c_{n-1}, \quad n = 1, 2, \ldots$

$$\boxed{c_n = \frac{1}{(n-1)n} c_{n-3} \quad , \quad n = 3, 4, \ldots}$$

$\underline{n = 3k}$: $\quad c_{3k} = \frac{1}{(3k-1) \cdot 3k} c_{3(k-1)} \quad , k \geq 1$

$k=1$: $\quad c_3 = \frac{1}{(3 \cdot 1 - 1) \cdot 3 \cdot 1} c_0$

$k=2$: $\quad c_6 = \frac{1}{(3 \cdot 2 - 1) \cdot 3 \cdot 2} c_3$

$\cdots\cdots$

$k=n$: $\quad c_{3n} = \frac{1}{(3n-1) 3n} c_{3(n-1)}$

$\Bigg\} \quad c_{3n} = \frac{1}{2 \cdot 5 \cdot 8 \cdots (3n-1) 3^n n!} c_0, \quad n \geq 1$

$\underline{n = 3k+2}$: $\quad c_{3k+2} = \frac{1}{(3k+1)(3k+2)} c_{3k-1} \quad , \quad k \geq 1$

$c_2 = 0 \Rightarrow c_5 = 0 \quad\quad c_{3k+2} = 0$

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} + \sum_{n=0}^{\infty} c_{3n+1} x^{3n+1} + \sum$$

Εξίσωση Legendre

$(1-x^2)y'' - 2xy' + p(p+1)y = 0 \quad , p\in\mathbb{R} \quad , x_0 = 0$

$$C_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)} C_n , \quad n = 0,1,\dots$$

$$\boxed{C_n = -\frac{(p-n+2)(p+n-1)}{(n-1)n} C_{n-2} , \quad n = 2,\dots}$$

$$C_{2n} = (-1)^n \frac{p(p-2)\cdots(p-2n+2)(p+1)\cdots(p+2n-1)}{(2n)!} c_0 \quad , n\geq 1$$

$$C_{2n+1} = (-1)^{n+1} \frac{(p-1)(p-3)\cdots(p-2n+1)(p+2)\cdots(p+2n)}{(2n+1)!} C_1 , \quad n\geq 1$$

$p\in\mathbb{N} \rightarrow$ πολυωνυμικές λύσεις $\rightarrow$ πολυώνυμα Legendre τάξης $\omega\in\mathbb{N}$: $P_\omega(x)$

Πρόταση: $\int_{-1}^{1} P_\omega(x) P_n(x)\, dx = 0 \quad , \forall \omega, n\in\mathbb{N} \quad , \omega\neq n.$

ΑΠΟΔ

$$\left.\begin{aligned} &[(1-x^2)y']' + p(p+1)y = 0 \\ &[(1-x^2)P_n']' + n(n+1)P_n = 0 \\ &[(1-x^2)P_\omega']' + \omega(\omega+1)P_\omega = 0 \end{aligned}\right\| \Rightarrow$$

$\Rightarrow [(1-x^2)P_n']P_\omega + n(n+1)P_nP_\omega - [(1-x^2)P_\omega']P_n - \omega(\omega+1)P_nP_\omega = 0$

$\int_{-1}^{1} [(1-x^2)P_n']' P_\omega - [(1-x^2)P_\omega']' P_n \, dx = \int_{-1}^{1}$